J. Math. Anal. Appl. 345 (2008) 203–212

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Estimates of positive linear operators in terms of second-order moduli

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ARTICLE INFO

Article history:
Received 17 September 2007
Available online 11 April 2008
Submitted by M. Milman

Keywords:
Moduli of smoothness
Positive linear operators
Bernstein operators
Best constants

ABSTRACT

We estimate the constants related with the direct result for positive linear operators which preserves linear functions. The estimates are presented for the modulus of smoothness $\omega_2^{\varphi}(f,h)$, where the weight φ is a function such that φ^2 is concave.

1. Introduction

Let $C[0,1]$ be the space of all real continuous functions on $[0,1]$ and $\Omega(0,1)$ the class of nonnegative functions $\varphi \in C[0,1]$ which are strictly positive on $(0,1)$, and such that φ^2 is concave.

If $\varphi \in \Omega(0,1)$ and $s > 0$ we define

$$I(\varphi,s) = \{x \in (0,1)\colon\ 0 \leqslant x - s\varphi(x) < x + s\varphi(x) \leqslant 1\},$$
$$I(\varphi) = \{s > 0\colon\ I(\varphi,s) \neq \emptyset\} \quad \text{and} \quad h_\varphi = \big(2\varphi(1/2)\big)^{-1}.$$

For $\varphi \in \Omega(0,1)$, $f \in C[0,1]$ and $h \in (0,h_\varphi]$ the weighted second-order modulus is defined by (Ditzian and Totik [3])

$$\omega_2^{\varphi}(f,h) = \sup_{0\leqslant s\leqslant h}\ \sup_{x\in I(\varphi,s)} \big|f\big(x-s\varphi(x)\big) - 2f(x) + f\big(x+s\varphi(x)\big)\big|. \tag{1}$$

The modulus (1) has been used to present estimates in approximation theory. Let us recall some of them. For $n \geqslant 1$ and $f \in C[0,1]$, the Bernstein operator B_n is defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1].$$

In [2], Ditzian proved that for $\alpha \in [0,1/2]$ and $\varphi(x) = (x(1-x))^{\alpha}$ there exists a constant C_ψ, such that, for $f \in C[0,1]$ and $x \in (0,1)$,

$$\big|f(x) - B_n(f,x)\big| \leqslant C_\varphi \omega_2^{\varphi}\left(f, \frac{\sqrt{x(1-x)}}{\sqrt{n}\varphi(x)}\right). \tag{2}$$

This result unifies the classical estimate for $\alpha = 0$ (Strukov and Timan [10]) with the norm estimate for $\alpha = 1/2$ (Ditzian and Totik [3, p. 117]). In [4] Felten proved (2) holds if $\varphi \in \Omega(0,1)$. On the other hand, in [6] Gavrea et al. verified that

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.04.017

$$\|f - B_n(f)\| \leqslant 3\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right), \tag{3}$$

for $\varphi(x) = \sqrt{x(1-x)}$. This last estimate improved some others given in [1,5,7]. In fact the main result of [6] provides an estimate for positive linear operators that preserve linear functions. The result was improved in [9]: if $L : C[0,1] \to C[0,1]$ is a positive linear operator, $f \in C[0,1]$, $0 < h \leqslant 1/2$ and $x \in (0,1)$, then

$$\left|f(x) - L(f,x)\right| \leqslant \left|f(x)\right|\left|1 - L(e_0, x)\right| + \frac{|L(e_1 - x, x)|}{h\varphi(x)}\omega_1^{\varphi}(f,h) + \left(1 + \frac{3}{2}\frac{L((e_1-x)^2, x)}{(h\varphi(x))^2}\right)\omega_2^{\varphi}(f,h),$$

where $\varphi(x) = \sqrt{x(1-x)}$.

In this paper we generalize the results of [6]. In particular we prove the estimations given in [6] for general weights $\varphi \in \Omega(0,1)$. This unified the results in [6] and the ones of Felten. We follow the method of proof presented in [6] and use some ideas of [4]. As a by product, we show that for $\varphi \in \Omega(0,1)$, $\lambda, t > 0$ such that $\lambda t \in (0, 1/2]$ and $f \in C[0,1]$,

$$\omega_2^{\varphi}(f, \lambda t) \leqslant (2 + 3\lambda^2)\omega_2^{\varphi}(f, t).$$

2. Concave functions

First we collect some properties of concave functions.

Notice that if φ^2 is concave, then φ is concave.

For $\varphi \in \Omega(0,1)$, $h \in (0, h_\varphi]$ and $x \in [0,1]$, define

$$F_\varphi(x) = x + h\varphi(x) \quad \text{and} \quad G_\varphi(x) = x - h\varphi(x),$$

and denote

$$\begin{aligned} &A(\varphi,h) = \{x \in (0,1]:\ h\varphi(x) < x\}, \qquad a_h = \inf(A(\varphi,h)),\\ &B(\varphi,h) = \{x \in [0,1):\ h\varphi(x) < 1 - x\} \quad \text{and} \quad b_h = \sup(B(\varphi,h)). \end{aligned} \tag{4}$$

It follows from Proposition 1 that $A(\varphi,h)$ and $B(\varphi,h)$ are intervals.

For $a \in [0,1)$ and $c \in (0,1]$ define

$$M_\varphi(a,y) = \frac{\varphi(y)}{y-a} \quad \text{and} \quad N_\varphi(c,z) = \frac{\varphi(z)}{c-z},$$

where $y \in (a,1]$ and $z \in [0,c)$.

Proposition 1. *Let $\varphi : [0,1] \to \mathbb{R}$ be a concave positive function.*

(i) *If $0 \leqslant a < 1$, then the function $M_\varphi(a,\circ)$ decreases on $(a,1]$.*
(ii) *If $0 < c \leqslant 1$, then $N_\varphi(c,\circ)$ increases on $[0,c)$. Moreover for $0 \leqslant a < c \leqslant 1$,*

$$\max\{\varphi(a), \varphi(c)\} \leqslant 2\varphi((a+c)/2). \tag{5}$$

(iii) *The limits*

$$\lim_{x\to 0}\frac{\varphi(x)}{x} \quad \textit{and} \quad \lim_{x\to 1}\frac{\varphi(x)}{1-x}$$

exist (finite or infinite).
(iv) *If $c - a \leqslant 2h\varphi((a+c)/2)$ and $a \leqslant u < v \leqslant c$, then $v - u \leqslant 2h\varphi((u+v)/2)$.*

Proof. Let us first verify a general inequality. Fix three numbers $0 \leqslant a_1 < a_2 < a_3 \leqslant 1$ and set $\tau = (a_2 - a_1)/(a_3 - a_1)$. Since φ is concave and positive,

$$\max\{\tau\varphi(a_3), (1-\tau)\varphi(a_1)\} \leqslant (1-\tau)\varphi(a_1) + \tau\varphi(a_3) \leqslant \varphi((1-\tau)a_1 + \tau a_3) = \varphi(a_2).$$

Thus

$$\frac{\varphi(a_1)}{a_3 - a_1} \leqslant \frac{\varphi(a_2)}{a_3 - a_2} \quad \text{and} \quad \frac{\varphi(a_3)}{a_3 - a_1} \leqslant \frac{\varphi(a_2)}{a_2 - a_1}. \tag{6}$$

(i) It follows from the second inequality in (6) with $a_1 = a$, that $M_\varphi(a,\circ)$ decreases on $(a,1]$.

(ii) It follows from the first inequality in (6) with $a_3 = c$, that $N_\varphi(c,y)$ increases on $[0,c)$. Moreover, if we take $a_1 = a$, $a_3 = c$ and $a_2 = (a+c)/2$ in (6) we obtain (5).

(iii) The assertions follow from (i) and (ii) by taking $a = 0$ and $c = 1$, respectively.

(iv) From (i) we obtain

$$\frac{2\varphi((u+v)/2)}{v-u} = N_\varphi\left(v, \frac{u+v}{2}\right) \geqslant N_\varphi\left(v, \frac{a+v}{2}\right) = M_\varphi\left(a, \frac{a+v}{2}\right) \geqslant M_\varphi\left(a, \frac{a+c}{2}\right) = \frac{2\varphi((a+c)/2)}{c-a} \geqslant \frac{1}{h}. \qquad \square$$

Proposition 2. *Fix $\varphi \in \Omega(0,1)$ and $h \in (0, h_\varphi]$.*

(i) *The function F_φ is strictly increasing on $[0, b_h]$, $F_\varphi(b_h) = 1$ and for $x \in [0, b_h)$ $([b_h, 1])$,*

$$0 \leqslant F_\varphi(x) < 1 \quad \bigl(1 \leqslant F_\varphi(x)\bigr). \tag{7}$$

(ii) *The function G_φ is strictly increasing on $[a_h, 1]$, $G_\varphi(a_h) = 0$ and for $x \in (a_h, 1]$ $([0, a_h])$,*

$$0 < G_\varphi(x) \leqslant 1 \quad \bigl(G_\varphi(x) \leqslant 0\bigr). \tag{8}$$

(iii) *For each $a \in [h\varphi(0), 1]$ and $b \in [0, 1 - h\varphi(1)]$, there exist unique points $x \in [0, b_h]$ and $y \in [a_h, 1]$ such that $F_\varphi(x) = a$ and $G_\varphi(y) = b$.*

(iv) *If $0 < s < h$, then $b_h \leqslant b_s$.*

Proof. Let us see that if $0 \leqslant x < y \leqslant b_h$, then $F_\varphi(x) < F_\varphi(y)$. Since $B(\varphi, h)$ is an interval and $x < b_h$, then $x + h\varphi(x) < 1$. Fix $\alpha \in [0,1]$ such that $y = \alpha x + (1-\alpha) b_h$. Since φ is concave, we have

$$\begin{aligned} x + h\varphi(x) &= \alpha\bigl(x + h\varphi(x)\bigr) + (1-\alpha)\bigl(x + h\varphi(x)\bigr) < \alpha\bigl(x + h\varphi(x)\bigr) + (1-\alpha) = \alpha\bigl(x + h\varphi(x)\bigr) + (1-\alpha)\bigl(b_h + h\varphi(b_h)\bigr) \\ &\leqslant \alpha x + (1-\alpha)b_h + h\varphi\bigl(\alpha x + (1-\alpha)b_h\bigr) = y + h\varphi(y). \end{aligned}$$

If $b_h < 1$, it follows from the continuity of φ that $1 - b_h = h\varphi(b_h)$. If $b_h = 1$, we obtain again the last equality. In fact,

$$\lim_{x \to 1} \frac{\varphi(x)}{1-x} \leqslant \frac{1}{h}.$$

Thus $\varphi(b_h) = 0$. In any case $F_\varphi(b_h) = 1$. For $x \in [0, b_h)$, taking into account that F_φ is increasing, $F_\varphi(x) < 1$.

We have proved (i). The assertion (ii) follows analogously. For instance, if $a_h \leqslant x < y$ and $x = \alpha a_h + (1-\alpha)y$, then

$$x - h\varphi(x) \leqslant (1-\alpha)\bigl(y - h\varphi(y)\bigr) + \alpha\bigl(a_h - h\varphi(a_h)\bigr) < y - h\varphi(y).$$

In order to verify (iii), fix $a \in [h\varphi(0), 1]$. Since $F_\varphi(0) = h\varphi(0) \leqslant a \leqslant 1 = F_\varphi(b_h)$, and F_φ is continuous and strictly increasing, the equation $F_\varphi(x) = a$ has one and only one solution. The other assertion follows analogously.

(iv) If $0 < s < h \leqslant 1/2$, since $A(h) \subset A(s)$, then $b_h \leqslant b_s$. $\square$

Definition 3. Fix $\varphi \in \Omega(0,1)$ and $h \in (0, h_\varphi)$. For each $x \in [0, b_h]$, the increasing chain $(\{y_n\}, \{z_n\})$ associated to (x, h) is defined as follows. Let $z_0 = x$, $y_1 = x + h\varphi(x)$. If $y_1 \geqslant 1 - h\varphi(1)$ the construction ends in y_1. If $y_1 < 1 - h\varphi(1)$, then $z_1 \in (y_1, 1)$ is the unique solution of the equation $y_1 = z_1 - h\varphi(z_1)$ (see (iii) in Proposition 2). For $j \geqslant 1$, if $z_j > b_h$, the construction ends in z_j. If $z_j \leqslant b_h$, we define $y_{j+1} = z_j + h\varphi(z_j)$. If $y_{j+1} \geqslant 1 - h\varphi(1)$ the construction ends. If $y_{n+1} < 1 - h\varphi(1)$, we define z_{j+1} as the unique solution of the equation $y_{j+1} = z_{j+1} - h\varphi(z_{j+1})$.

The greater integer q, such that y_q is well defined, is called the length of the chain (we will see bellow that q is a finite number).

Similarly, for $a_h < x \leqslant 1$, we define the decreasing chain $(\{y_n^*\}, \{z_n^*\})$ associated to (x, h), such that $z_0^* = x$, $y_j^* = z_j^* - h\varphi(z_j^*)$ and $z_{j+1}^* + h\varphi(z_{j+1}^*) = y_j^*$.

Proposition 4. *Fix $\varphi \in \Omega(0,1)$ and $h \in (0, h_\varphi)$ and let b_h be given by (4). For $x \in (0, b_h)$ let $(\{y_n\}, \{z_n\})$ be the increasing chain associated to (x, h).*

(i) *The set $U(x,h)$ of all integers q, such that y_q is defined, is finite.*

(ii) *If $p = \max U(x,h)$, then $y_p \geqslant 1 - h\varphi(1)$ or $y_p < 1 - h\varphi(1)$ and $b_h < z_p$.*

(iii) *If $t \in (y_1, 1]$ and $z_1 > b_h$, then*

$$t - y_1 \leqslant 2h\varphi\left(\frac{t + y_1}{2}\right). \tag{9}$$

The same estimate holds if $z_1 \leqslant b_h$ and $t \leqslant y_2$.

Proof. Since φ^2 is concave, one has $b_h < 1$.

(i) Suppose that $U(x,h)$ is an infinite set. Then, for all $n > q$, $z_n \leqslant b_h$. Thus

$$z_{j+q} = y_{q+j-1} + h\varphi(z_{j+q}) = z_{j-1+q} + h\big(\varphi(z_{j-1+q}) + \varphi(z_{j+q})\big) \geqslant z_q + h\sum_{i=0}^{j}\varphi(z_{i+q}) = z_q + h\sum_{i=0}^{j}(1 - z_{i+q})\frac{\varphi(z_{i+q})}{1 - z_{i+q}}$$

$$\geqslant z_q + h\frac{\varphi(z_q)}{1 - z_q}\sum_{i=0}^{j}(1 - z_{i+q}) \geqslant z_q + h\frac{\varphi(z_q)}{1 - z_q}(j+1)(1 - b_h)$$

and we obtain a contradiction.

(ii) It follows from the definition of p.

(iii) *Case* 1. Assume $z_1 > b_h$. For $s \in [y_1, 1]$ the function $P(s) = 2h\varphi((s + y_1)/2) - s + y_1$ is concave. It is sufficient to show that $P(y_1) \geqslant 0$ and $P(1) \geqslant 0$. The first assertion is evident. On the other hand, since $b_h < z_1$, taking into account Proposition 2,

$$b_h = \frac{1 + G_\varphi(b_h)}{2} \leqslant \frac{1 + G_\varphi(z_1)}{2} = \frac{1 + y_1}{2}.$$

Since $N_\varphi(1, \circ)$ is an increasing function

$$\frac{2\varphi((1 + y_1)/2)}{1 - y_1} = N_\varphi\left(1, \frac{1 + y_1}{2}\right) \geqslant N_\varphi(1, b_h) = \frac{\varphi(b_h)}{1 - b_h} = h.$$

This proves $P(1) \geqslant 0$.

Case 2. Assume that $z_1 \leqslant b_h$ and $t < y_2$ (y_2 is well defined in this case). Since $y_1 < t \leqslant y_2$ and $y_2 - y_1 = 2h\varphi((y_1 + y_2)/2)$, the estimate follows from (iii) in Proposition 1. □

3. Auxiliary results

If $f : [0,1] \to \mathbb{R}$, $a, x, c \in [0,1]$ and $a \neq b$ we denote

$$\Delta(f, a, x, b) = \frac{b - x}{b - a}f(a) + \frac{x - a}{b - a}f(b) - f(x).$$

Proposition 5. *Let $\varphi \in \Omega(0,1)$, $0 \leqslant a < b \leqslant 1$, $c = (a+b)/2$ and $x \in [a,b]$. If $f \in C[0,1]$, then*

$$\big|\Delta(f, a, c, b)\big| \leqslant \frac{1}{2}\omega_2^\varphi\left(f, \frac{b - a}{2\varphi(c)}\right) \tag{10}$$

and

$$\big|\Delta(f, a, x, b)\big| \leqslant \omega_2^\varphi\left(f, \frac{b - a}{2\varphi(c)}\right). \tag{11}$$

Proof. Set $g(x) = \Delta(f, a, x, b)$, $x \in [0,1]$. We get $g(a) = g(b) = 0$ and $\omega_2^\varphi(g, t) = \omega_2^\varphi(f, t)$. The first assertion follows from the identity

$$g(c) = -\frac{1}{2}\big(g(a) - 2g(c) + g(b)\big).$$

Let M be the supremum of $|g(t)|$ for $t \in [a,b]$. Fix $u \in (a,b)$ such that $|g(u)| = M$. We can assume $g(u) > 0$ and $u \geqslant c$. Since (see Proposition 1)

$$\frac{2\varphi(c)}{b - a} = N_\varphi(b, c) \leqslant N_\varphi(b, u) = \frac{\varphi(u)}{b - u},$$

one has

$$M = g(u) = -g(b) + 2g(u) - g(2u - b) - M + g(2u - b) \leqslant \omega_2^\varphi\left(g, \frac{b - u}{\varphi(u)}\right) \leqslant \omega_2^\varphi\left(g, \frac{b - a}{2\varphi((a+b)/2)}\right). \qquad \square$$

Proposition 6. *Fix $\varphi \in \Omega(0,1)$. Let $h \in (0, h_\varphi)$ and $x \in [a_h, b_h]$ be such that the increasing chain associated to (x, h) has length* 1 *and $y_1 < 1 - h\varphi(1)$. If $f \in C[0,1]$ and $f(x - h\varphi(x)) = 0 = f(x + h\varphi(x))$, then for $t \in [y_1, 1]$,*

$$\big|f(t)\big| \leqslant \left(\frac{1}{2} + \frac{3}{2}\left(\frac{t - x}{h\varphi(x)}\right)^2\right)\omega_2^\varphi(f, h).$$

The same estimation holds if y_2 exists and $t \in [y_1, y_2]$.

Proof. *Case* 1. Assume $t \leqslant z_1$. Since $t \in [x, z_1]$ and

$$z_1 - x = h\big(\varphi(x) + \varphi(z_1)\big) \leqslant 2h\varphi\left(\frac{x+z_1}{2}\right), \tag{12}$$

it follows from (iii) in Proposition 1, (10) and (11) that

$$\begin{aligned}\big|f(t)\big| &= \left|\frac{t-x}{y_1-x}\Delta(f,x,y_1,t) - \frac{t-y_1}{y_1-x}f(x)\right| \leqslant \left(\frac{t-x}{y_1-x} + \frac{1}{2}\frac{t-y_1}{y_1-x}\right)\omega_2^{\varphi}(f,h) = \left(-\frac{1}{2} + \frac{3}{2}\frac{t-x}{y_1-x}\right)\omega_2^{\varphi}(f,h)\\ &\leqslant \left(\frac{1}{2} + \frac{3}{2}\left(\frac{t-x}{y_1-x}\right)^2\right)\omega_2^{\varphi}(f,h).\end{aligned} \tag{13}$$

Case 2. Assume $z_1 < t$ and $x + (4/3)h\varphi(x) \leqslant t$. If we denote $a = y_1 - x$, $b = z_1 - y_1$ and $c = t - z_1$, the last condition can be rewritten as $4a \leqslant 3(a+b+c)$. Since φ^2 is concave and $N_{\varphi^2}(t,\circ)$ is increasing,

$$ac = \frac{c}{b}\left(\frac{ab^2}{a+b} + \frac{ba^2}{a+b}\right) \leqslant h^2\frac{c}{b}\varphi^2\left(\frac{a}{a+b}z_1 + \frac{b}{a+b}x\right) = h^2\frac{c}{b}\varphi^2(y_1) \leqslant h^2\frac{c}{b}\varphi^2(z_1)\frac{t-y_1}{t-z_1} = \frac{1}{b}b^2(b+c) = b(b+c). \tag{14}$$

Therefore

$$4a^2c \leqslant 4ab(b+c) \leqslant 3b(b+c)(a+b+c),$$

which is equivalent to the inequality

$$2\frac{b+c}{b} + \frac{3}{2}\frac{b+c}{a} \leqslant \frac{1}{2} + \frac{3}{2}\left(1 + \frac{b+c}{a}\right)^2.$$

Thus, taking into account (9), (12) and the identity

$$f(t) = \frac{t-y_1}{h\varphi(z_1)}\Delta(f,y_1,z_1,t) - \frac{t-y_1}{h\varphi(z_1)}\frac{z_1-x}{y_1-x}\Delta(f,x,y_1,z_1) - \frac{t-y_1}{y_1-x}f(x),$$

we obtain

$$\begin{aligned}\big|f(t)\big| &\leqslant \left(\frac{t-y_1}{h\varphi(z_1)} + \frac{t-y_1}{h\varphi(z_1)}\frac{z_1-x}{y_1-x} + \frac{1}{2}\frac{t-y_1}{y_1-x}\right)\omega_2^{\varphi}(f,h) = \left(2\frac{t-y_1}{z_1-y_1} + \frac{3}{2}\frac{t-y_1}{y_1-x}\right)\omega_2^{\varphi}(f,h)\\ &= \left(2\frac{b+c}{b} + \frac{3}{2}\frac{b+c}{a}\right)\omega_2^{\varphi}(f,h) \leqslant \left(\frac{1}{2} + \frac{3}{2}\left(\frac{t-x}{y_1-x}\right)^2\right)\omega_2^{\varphi}(f,h).\end{aligned}$$

Case 3. Assume $z_1 < t < x + (4/3)h\varphi(x)$. Let us first prove the inequality

$$t < y_1 + h\varphi(y_1). \tag{15}$$

We assume a, b and c are defined as in Case 2. Thus $3(b+c) < a$.

Since $b_h < z_1$, from (7) we get $y_1 + 2h\varphi(z_1) = z_1 + h\varphi(z_1) \geqslant 1$. Therefore $t + y_1 \leqslant 1 + y_1 \leqslant 2(y_1 + h\varphi(z_1)) = 2z_1$. That is

$$c = t - z_1 \leqslant z_1 - y_1 = b.$$

From this last inequality we obtain (see (14))

$$(t-y_1)^2 = (b+c)^2 < \frac{1}{3}a(b+c) \leqslant \frac{2}{3}ab \leqslant \frac{2}{3}\frac{c}{b}h^2\varphi^2(y_1)$$

and this proves (15).

Since $t \leqslant x + (4/3)h\varphi(x)$, we have $x - h\varphi(x) \leqslant 2y_1 - t = y_1 - (t - y_1) \leqslant y_1$. Thus

$$\big|f(t)\big| = \big|f(t) - 2f(y_1) + f(2y_1 - t) + f(2y_1 - t)\big| \leqslant 2\omega_2^{\varphi}(f,h)$$

and

$$2 \leqslant \frac{1}{2} + \frac{3}{2}\left(\frac{t-x}{y_1-x}\right)^2.$$

Finally, we prove the last assertion. Assume y_2 exists and $t \in [y_1, y_2]$. If $t \leqslant z_1$ the proof follows as in Case 1. If $z_1 < t \leqslant y_2$ and $x + (4/3)h\varphi(x) \leqslant t$, we can repeat the arguments of Case 2. Notice that $y_2 = y_1 + 2h\varphi(z_1)$, thus $|\Delta(f, y_1, z_1, t)| \leqslant \omega_2^{\varphi}(f,h)$. Finally, if $z_1 < t < x + (4/3)h\varphi(x)$, we shall modify the proof of (15). In this case

$$t + y_1 \leqslant y_1 + y_2 = 2\big(y_1 + h\varphi(z_1)\big) = 2z_1,$$

and the rest of the proof follows as in Case 3. $\square$

Proposition 7. *Fix $\varphi \in \Omega(0,1)$. Let $h \in (0, h_\varphi)$ and $x \in [a_h, b_h]$ be such that the increasing chain associated to (x,h) has length 1 and $y_1 \geqslant 1 - h\varphi(1)$. If $f \in C[0,1]$ and $f(x - h\varphi(x)) = 0 = f(x + h\varphi(x))$, then for $t \in [x + h\varphi(x), 1]$,*

$$\left|f(t)\right| \leqslant \left(\frac{1}{2} + \frac{3}{2}\left(\frac{t-x}{h\varphi(x)}\right)^2\right)\omega_2^\varphi(f,h).$$

Proof. Let us denote $d = (x + y_1)/2$.

Case 1. Assume $b_h \leqslant d$. Notice that $(y_1 - x) \leqslant 2(y_1 - b_h)$ and $b_h < y_1$. From (5), $\varphi(b_h) \leqslant 2\varphi((1 + b_h)/2)$ and it follows from (iii) in Proposition 1 that $t - b_h \leqslant 2h\varphi((t + b_h)/2$. Then

$$\begin{aligned}\left|f(t)\right| &= \left|\frac{t-b_h}{y_1-b_h}\Delta(f, b_h, y_1, t) - \frac{t-y_1}{y_1-b_h}f(b_h)\right| \leqslant \frac{1}{y_1-b_h}(2t - b_h - y_1)\omega_2^\varphi(f,h) = \left(1 + 2\frac{t-y_1}{y_1-b_h}\right)\omega_2^\varphi(f,h)\\ &\leqslant \left(1 + 4\frac{t-y_1}{y_1-x}\right)\omega_2^\varphi(f,h) = \left(-3 + 4\frac{t-x}{y_1-x}\right)\omega_2^\varphi(f,h) \leqslant \left(\frac{1}{2} + \frac{3}{2}\left(\frac{t-x}{h\varphi(x)}\right)^2\right)\omega_2^\varphi(f,h),\end{aligned}$$

since for $v \in \mathbb{R}$, $3v^2 - 8v + 7 \geqslant 0$.

Case 2. Assume $d < b_h \leqslant y_1$. Since $\varphi(x) \leqslant 2\varphi(d)$ (see (5)) we have

$$\begin{aligned}t - d &= t - h\varphi(t) + h\varphi(t) - d \leqslant 1 - h\varphi(1) + h\varphi(t) - d \leqslant y_1 + h\varphi(t) - d = \frac{h\varphi(x)}{2} + h\varphi(t)\\ &\leqslant h\big(\varphi(d) + \varphi(t)\big) \leqslant 2h\varphi\big((t+d)/2\big)\end{aligned} \tag{16}$$

and

$$\frac{1}{y_1-d}(t - d + t - y_1) = 1 + 2\frac{t-y_1}{y_1-d} = 1 + 4\frac{t-y_1}{y_1-x} = -3 + 4\frac{t-x}{y_1-x}.$$

Therefore

$$\left|f(t)\right| \leqslant \left|\frac{t-d}{y_1-d}\Delta(f, d, y_1, t) - \frac{t-y_1}{y_1-d}f(d)\right| \leqslant \frac{1}{y_1-d}(2t - d - y_1)\omega_2^\varphi(f,h),$$

and the estimate follows from the arguments given in Case 1.

Case 3. Assume $y_1 < b_h$ and $t \leqslant y_1 + h\varphi(y_1)$. Since $y_1 + h\varphi(y_1) < 1$ (see (7)),

$$1 - h\varphi(1) \leqslant y_1 < 1 - h\varphi(y_1).$$

Thus $\varphi(y_1) < \varphi(1)$. Since φ is concave, $\varphi(x) \leqslant \varphi(y_1)$. Therefore

$$y_1 - h\varphi(y_1) \leqslant x < y_1 \leqslant t \leqslant y_1 + h\varphi(y_1),$$

and it follows from (iv) in Proposition 1 (with $a = y_1 - h\varphi(y_1)$, $b = y_1 + h\varphi(y_1)$, $u = x$ and $v = t$) that $t - x \leqslant 2h\varphi((x+t)/2)$. Moreover, from (11) we obtain

$$\left|\Delta(f, x, y_1, t)\right| \leqslant \omega_2^\varphi(f,h).$$

From the argument given above it follows

$$\begin{aligned}\left|f(t)\right| &= \left|\frac{t-x}{y_1-x}\Delta(f, x, y_1, t) - \frac{t-y_1}{y_1-x}f(x)\right| \leqslant \left(\frac{t-x}{y_1-x} + \frac{1}{2}\frac{t-y_1}{y_1-x}\right)\omega_2^\varphi(f,h) = \left(-\frac{1}{2} + \frac{3}{2}\frac{t-x}{y_1-x}\right)\omega_2^\varphi(f,h)\\ &\leqslant \left(\frac{1}{2} + \frac{3}{2}\left(\frac{t-x}{h\varphi(x)}\right)^2\right)\omega_2^\varphi(f,h),\end{aligned}$$

since for $v \in \mathbb{R}$, $3v^2 - 3v + 2 \geqslant 0$.

Case 4. Finally, assume $y_1 < b_h$ and $y_1 + h\varphi(y_1) < t$. As in Case 3 we have $\varphi(x) \leqslant \varphi(y_1)$. Thus

$$\frac{t-x}{y_1-x} \geqslant \frac{y_1 + h\varphi(y_1) - x}{h\varphi(x)} = 1 + \frac{\varphi(y_1)}{\varphi(x)} \geqslant 2.$$

On the other hand, from (5) we have

$$t - y_1 = t - h\varphi(t) + h\varphi(t) - y_1 \leqslant 1 - h\varphi(1) + h\varphi(t) - y_1 \leqslant y_1 + h\varphi(t) - y_1 = h\varphi(t) \leqslant 2h\varphi\left(\frac{t+y_1}{2}\right).$$

Since $N_t(y_1) \geqslant N_t(x)$, by considering the identity

$$f(t) = \frac{t-y_1}{h\varphi(y_1)}\Delta\big(f, y_1, y_1 + h\varphi(y_1), t\big) + \frac{t-y_1}{h\varphi(y_1)}\left(\frac{y_1 + h\varphi(y_1) - x}{y_1 - x}\Delta\big(f, x, y_1, y_1 + h\varphi(y_1)\big) - \frac{h\varphi(y_1)}{y_1-x}f(x)\right),$$

we have

$$\begin{aligned}
\left|f(t)\right| &\leqslant \frac{t-y_1}{h\varphi(y_1)}\left(1+\frac{y_1+h\varphi(y_1)-x}{y_1-x}+\frac{1}{2}\frac{h\varphi(y_1)}{y_1-x}\right)\omega_2^{\varphi}(f,h)=\left(2\frac{t-y_1}{h\varphi(y_1)}+\frac{3}{2}\frac{t-y_1}{y_1-x}\right)\omega_2^{\varphi}(f,h)\\
&\leqslant \left(-\frac{3}{2}+\frac{7}{2}\frac{t-x}{h\varphi(x)}\right)\omega_2^{\varphi}(f,h),
\end{aligned}$$

and the result follows because $3\nu^2-7\nu+4\geqslant 0$, for $\nu\geqslant 2$. $\square$

Proposition 8. *Fix $\varphi\in\Omega(0,1)$, $h\in(0,h_\varphi)$, $x\in[a_h,b_h]$ and $t\in[0,1]$ such that $0\leqslant t\leqslant x-h\varphi(x)$ or $x+h\varphi(x)\leqslant t\leqslant 1$. If $f\in C[0,1]$ satisfies $f(x-h\varphi(x))=0=f(x+h\varphi(x))$, then*

$$\left|f(t)\right|\leqslant\left(\frac{1}{2}+\frac{3}{2}\left(\frac{t-x}{h\varphi(x)}\right)^2\right)\omega_2^{\varphi}(f,h). \tag{17}$$

Proof. We consider $x+h\varphi(x)\leqslant t$ and present a proof by induction with respect to the length of the chains. That is, we consider the sequence of propositions (P_n), where (P_n) states that, for any pair (x,h), $a_h\leqslant x\leqslant b_h$, with increasing chain of length n, any $f\in C[0,1]$ such that $f(x-h\varphi(x))=0=f(x+h\varphi(x))$ and each $t\in(x+h\varphi(x),1]$, one has (17) (the case $t=x+h\varphi(x)$ is trivial).

Case $n=1$. If $(\{y_n\},\{z_n\})$ is the increasing chain of (x,h) and has length 1, then $y_1\geqslant 1-h\varphi(1)$ or $y_1<1-h\varphi(1)$ and $b_h<z_1$. In any case the estimate follows from Propositions 6 and 7.

Assume that (P_n) holds and fix a pair (x,h) ($a_h\leqslant x\leqslant b_h$) with increasing chain $(\{y_n\},\{z_n\})$ of length $n+1$. If we eliminate from the chain the points z_0 and y_1, we obtain the increasing chain associated to (z_1,h) (with length n). Fix $f\in C[0,1]$ and $t>x+h\varphi(x)$. If $y_1<t\leqslant z_1$ the estimate follows as in Case 1 in Proposition 6 (notice that in proving Case 1 in Proposition 6 we only need that z_1 is well defined). Thus we assume $t>z_1$. Let P be the polynomial of degree not greater than 1 which interpolates f at y_1 and y_2 and define $g=f-P$. Notice that $\omega_2^{\varphi}(f,h)=\omega_2^{\varphi}(g,h)$, $g(y_1)=g(y_2)=0$ and $\Delta(f,y_1,y_2,t)=\Delta(g,y_1,y_2,t)$. From the hypothesis of induction (for (z_1,h)) we obtain

$$\left|g(t)\right|\leqslant\left(\frac{1}{2}+\frac{3}{2}\left(\frac{t-z_1}{z_1-y_1}\right)^2\right)\omega_2^{\varphi}(f,h).$$

Taking into account the identity

$$f(y_2)=\frac{y_2-y_1}{z_1-y_1}\left(\Delta(f,y_1,z_1,y_2)+\frac{z_1-x}{y_1-x}\Delta(f,x,y_1,z_1)-\frac{z_1-y_1}{y_1-x}f(x)\right)$$

and using (10), we obtain

$$\begin{aligned}
\left|f(t)\right| &= \left|\frac{t-y_1}{y_2-y_1}\Delta(f,y_1,y_2,t)+\frac{t-y_1}{y_2-y_1}f(y_2)\right|=\left|g(t)+\frac{t-y_1}{y_2-y_1}f(y_2)\right|\\
&\leqslant \left|g(t)\right|+\frac{t-y_1}{y_2-y_1}\frac{y_2-y_1}{z_1-y_1}\left(\frac{1}{2}+\frac{z_1-x}{y_1-x}+\frac{1}{2}\frac{z_1-y_1}{y_1-x}\right)\omega_2^{\varphi}(f,h)\\
&\leqslant \left(\frac{1}{2}+\frac{3}{2}\left(\frac{t-z_1}{z_1-y_1}\right)^2+3\frac{t-y_1}{y_2-y_1}\frac{z_1-x}{y_1-x}\right)\omega_2^{\varphi}(f,h).
\end{aligned}$$

We finish the proof by showing that, for $t\in(z_1,1]$, $\Lambda(t)\leqslant 0$, where

$$\Lambda(t)=\left(\frac{t-z_1}{z_1-y_1}\right)^2-\left(\frac{t-x}{y_1-x}\right)^2+2\frac{t-y_1}{y_2-y_1}\frac{z_1-x}{y_1-x}.$$

Since $y_2-y_1=2(z_1-y_1)$,

$$\begin{aligned}
\Lambda(t) &= \left(\frac{t-z_1}{z_1-y_1}\right)^2-\left(\frac{t-x}{y_1-x}\right)^2+\frac{t-y_1}{z_1-y_1}\frac{z_1-x}{y_1-x}\\
&= \frac{(t-y_1)(z_1-x)}{(z_1-y_1)(y_1-x)}\left(\frac{(t-y_1)(2y_1-x-z_1)}{(z_1-y_1)(y_1-x)}-2\right)+\frac{t-y_1}{z_1-y_1}\frac{z_1-x}{y_1-x}\\
&= \frac{(t-y_1)(z_1-x)}{(z_1-y_1)(y_1-x)}\left(\frac{(t-y_1)(2y_1-x-z_1)}{(z_1-y_1)(y_1-x)}-1\right).
\end{aligned}$$

Thus it is sufficient to show

$$(1-y_1)(2y_1-x-z_1)\leqslant(z_1-y_1)(y_1-x),$$

or

$$(1-y_1)\big(\varphi(x)-\varphi(z_1)\big) \leqslant h\varphi(x)\varphi(z_1).$$

Since

$$(1-z_1)\varphi^2(x) \leqslant (1-x)\varphi^2(z_1),$$

then

$$h = \frac{z_1-x}{\varphi(x)+\varphi(z_1)} \leqslant \frac{(1-x)\varphi(z_1)-(1-z_1)\varphi(x)}{\varphi(x)\varphi(z_1)} = \frac{1-x}{\varphi(x)} - \frac{1-z_1}{\varphi(z_1)}.$$

Therefore

$$(1-y_1)\frac{\varphi(x)-\varphi(z_1)}{\varphi(x)\varphi(z_1)} = \frac{1-z_1+h\varphi(z_1)}{\varphi(z_1)} - \frac{1-x-h\varphi(x)}{\varphi(x)} = 2h + \frac{1-z_1}{\varphi(z_1)} - \frac{1-x}{\varphi(x)} \leqslant h. \qquad \square$$

4. Main results

In [3, Theorem 4.1.2] it was proved that if $\varphi \in \Omega(0,1)$, there exist constants C_φ and t_0 such that, if $f \in C[0,1]$, n is a positive integer and $nt \in (0, t_0]$, then

$$\omega_2^\varphi(f, nt) \leqslant C_\varphi n^2 \omega_2^\varphi(f,t).$$

By Proposition 8 this inequality can be improved as follows.

Theorem 9. *Fix $\varphi \in \Omega(0,1)$. If $f \in C[0,1]$, λ is a positive real and $\lambda t \in (0, h_\varphi)$, then*

$$\omega_2^\varphi(f, \lambda t) \leqslant \big(2+3\lambda^2\big)\omega_2^\varphi(f,t).$$

Proof. We can consider $\lambda > 1$. Fix $a \in (0,1)$ and $s \in (0, \lambda t]$ such that $\omega_2^\varphi(f,\lambda t) = |f(a-s\varphi(a)) - 2f(a) + f(a+s\varphi(a))|$. It is sufficient to consider the case $t < s$. Let P be the polynomial of degree not greater than 1 which interpolates f at $a - t\varphi(a)$ and $a + t\varphi(a)$ and set $g = f - P$. We have

$$\begin{aligned}\omega_2^\varphi(f,\lambda t) &= \big|g\big(a-s\varphi(a)\big) - 2g(a) + g\big(a+s\varphi(a)\big)\big| \\ &\leqslant \big|g\big(a-s\varphi(a)\big)\big| + \big|g\big(a+s\varphi(a)\big)\big| + \big|g\big(a-t\varphi(a)\big) - 2g(a) + g\big(a+t\varphi(a)\big)\big| \\ &\leqslant \left(2 + \frac{3}{2}\left(\frac{a-(a-s\varphi(a))}{t\varphi(a)}\right)^2 + \frac{3}{2}\left(\frac{a+s\varphi(a)-a}{t\varphi(a)}\right)^2\right)\omega_2^\varphi(f,t) \leqslant \big(2+3\lambda^2\big)\omega_2^\varphi(f,t). \qquad \square\end{aligned}$$

Theorem 10. *Fix $\varphi \in \Omega(0,1)$. If $f \in C[0,1]$ and $0 \leqslant t_1 < x < t_2 \leqslant 1$, then*

$$\big|\Delta(f,t_1,x,t_2)\big| \leqslant \left(\frac{3}{2} + \frac{3}{2h^2}\frac{(t_2-x)(x-t_1)}{\varphi^2(x)}\right)\omega_2^\varphi(f,h).$$

Proof. Define s, p and c by

$$x - t_1 = sh\varphi(x), \qquad t_2 - x = ph\varphi(x) \quad \text{and} \quad c = \frac{t_1+t_2}{2}.$$

We give a proof for $0 < s \leqslant p$.

If $s < 1$ and

$$h < \frac{t_2-t_1}{2\varphi(c)}, \tag{18}$$

then $p \geqslant 1$. In fact if $p < 1$, then $s + p < 2$ and $[t_1, t_2] \subset [x - ph\varphi(x), x + ph\varphi(x)]$. It follows from (iii) in Proposition 1 that $(t_2 - t_1) \leqslant 2h\varphi(c)$. But this contradicts (18).

We will consider 3 cases.

Case 1. If $(t_2 - t_1) \leqslant 2h\varphi(c)$, then (see (11))

$$\big|\Delta(f,t_1,x,t_2)\big| \leqslant \omega_2^\varphi\left(f, \frac{t_2-t_1}{2\varphi(c)}\right) \leqslant \omega_2^\varphi(f,h).$$

Case 2. If $(t_2 - t_1) > 2h\varphi(c)$ and $p \geqslant s \geqslant 1$, we can assume $f(x - h\varphi(x)) = f(x + h\varphi(x)) = 0$. It follows from Proposition 8 that

$$\left|f(t_1)\right| \leqslant \left(\frac{1}{2}+\frac{3}{2}s^2\right)\omega_2^{\varphi}(f,h) \quad \text{and} \quad \left|f(t_2)\right| \leqslant \left(\frac{1}{2}+\frac{3}{2}p^2\right)\omega_2^{\varphi}(f,h).$$

Now

$$\left|\Delta(f,t_1,x,t_2)\right| = \left|\frac{p}{s+p}f(t_1)+\frac{s}{s+p}f(t_2)-f(x)\right| \leqslant \left(1+\frac{3}{2}\frac{ps^2+sp^2}{s+p}\right)\omega_2^{\varphi}(f,h) = \left(1+\frac{3}{2}ps\right)\omega_2^{\varphi}(f,h).$$

Case 3. $(t_2-t_1) > 2h\varphi(c)$ and $s<1$. We first prove $x - sh\varphi(x) < b_h - h\varphi(b_h)$. In fact if $x - sh\varphi(x) \geqslant b_h - h\varphi(b_h)$, since

$$b_h - h\varphi(b_h) \leqslant x - sh\varphi(x) = t_1 < x < t_2 \leqslant 1 = b_h + h\varphi(b_h),$$

it follows from (iii) in Proposition 3 (with $u=t_1$, $v=t_2$, $c=1$ and $a=b_h-h\varphi(b_h)$) that $t_2-t_1 \leqslant 2h\varphi((t_1+t_2)/2) = 2h\varphi(c)$, and we have a contradiction.

Since $b_h - h\varphi(b_h) \leqslant 1 - h\varphi(1)$, there exists y such that $t_1 = y - h\varphi(y)$. It is clear that $x<y<b_h$. Now we can assume $f(y-h\varphi(y)) = f(y+h\varphi(y)) = 0$. Since $s<1$, then $p \geqslant 1$ (see the remark at the beginning of the proof). Thus, if $t_2 \leqslant y + h\varphi(y)$, it follows from (11)

$$\left|f(t_2)\right| \leqslant \omega_2^{\varphi}(f,h) \leqslant \left(\frac{1}{2}+\frac{3}{2}p^2\right)\omega_2^{\varphi}(f,h).$$

If $t_2 > y + h\varphi(y)$ a similar estimate follows from Proposition 8. In fact

$$\left|f(t_2)\right| \leqslant \left(\frac{1}{2}+\frac{3}{2}\left(\frac{t_2-y}{h\varphi(y)}\right)^2\right)\omega_2^{\varphi}(f,h) \leqslant \left(\frac{1}{2}+\frac{3}{2}\left(\frac{t_2-x}{h\varphi(x)}\right)^2\right)\omega_2^{\varphi}(f,h),$$

because the function $N_\varphi(t_2,\circ)$ increases on $[0,t_2)$. Therefore

$$\left|\Delta(f,t_1,x,t_2)\right| = \left|\frac{s}{s+p}f(t_2)-f(x)\right| \leqslant \left(1+\frac{s}{s+p}\left(\frac{1}{2}+\frac{3}{2}p^2\right)\right)\omega_2^{\varphi}(f,h) \leqslant \left(\frac{3}{2}+\frac{3}{2}sp\right)\omega_2^{\varphi}(f,h). \qquad \square$$

As usually, we denote $e_0(t)=1$, $e_1(t)=t$ and $e_2(t)=t^2$. Thus, if L is a linear operator which reproduces linear functions, then

$$L\big((e_1-xe_0)^2,x\big) = L(e_2,x) - 2xL(e_1,x) + x^2 = L(e_2,x) - x^2.$$

Theorem 11. *Fix $\varphi \in \Omega(0,1)$. Let $L: C[0,1] \to C[0,1]$ be a positive linear operator such that $L(e_0)=e_0$ and $L(e_1)=e_1$. For $f \in C[0,1]$ and $x \in (0,1)$, one has*

$$\left|f(x)-L(f,x)\right| \leqslant \left(\frac{3}{2}+\frac{3}{2(h\varphi(x))^2}\big(L(e_2,x)-x^2\big)\right)\omega_2^{\varphi}(f,h).$$

Proof. If we set

$$\psi(t) = \frac{3}{2}+\frac{3t^2}{2\varphi^2(x)},$$

the result in Theorem 10 can be rewritten as

$$\left|\Delta(f,t_1,x,t_2)\right| \leqslant \left(\frac{t_2-x}{t_2-t_1}\psi\left(\frac{x-t_1}{h}\right)+\frac{x-t_1}{t_2-t_1}\psi\left(\frac{t_2-x}{h}\right)\right)\omega_2^{\varphi}(f,h).$$

Then, it follows from Corollary 1.1 of [8] that

$$\left|f(x)-L(f,x)\right| \leqslant L\left(\psi\left(\left|\frac{e_1-xe_0}{h}\right|\right),x\right)\omega_2^{\varphi}(f,h) = \left(\frac{3}{2}+\frac{3}{2h^2\varphi^2(x)}L\big((e_1-xe_0)^2,x\big)\right)\omega_2^{\varphi}(f,h). \qquad \square$$

Theorem 12. *Fix $\varphi \in \Omega(0,1)$. For $n \geqslant 1$, $f \in C[0,1]$ and $x \in (0,1)$ one has*

$$\left|f(x)-B_n(f,x)\right| \leqslant 3\omega_2^{\varphi}\left(f,\frac{\sqrt{x(1-x)}}{\sqrt{n}\varphi(x)}\right),$$

where B_n is the Bernstein operator.

Proof. Recall that $B_n((e_1-xe_0)^2;x) = x(1-x)/n$. Thus the result follows from Theorem 11 with $h = \sqrt{x(1-x)}/(\varphi(x)\sqrt{n})$. $\square$

Remark 13. If $\varphi(x) = (x(1-x))^{\alpha}$, with $\alpha \in (0,1/2]$, then Theorem 12 holds.

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